DYNAMICS OF THE VIBRATION OF A BAR EXCITED BY THE LONGITUDINAL IMPACT OF AN ELASTIC LOAD

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ABSTRACT. Dynamics of the vibration of a bar excited by longitudinal impact by an elastic load has been worked out, following operational method. The main idea upon which the dynamics is built up is that the bar behaves like a loaded bar so long the load is in contact with it, and the elastic load is supposed to behave like a hard load backed by a weightless spring. Two distinct cases have been worked out from the general solution, namely, (i) when the load strikes the free end, and (ii) when the load strikes the fixed end of the bar. The method gives the solution for any epoch very easily, unlike the method of the variation of intigration constant where solution for any particular epoch can be obtained after a long laborious calculation.

INTRODUCTION

The theory of the extensional vibration of a bar excited by the impact of a rigid load has been worked out by a number of workers (Boussinesq, 1885). Ghosh (1935) extended the case, applying the same method, for an elastic load struck at the free end of a bar, the other end being fixed. Later Ghosh and Dhar (1930) worked out the case for a bar struck at the fixed end by an elastic load, the other end remaining free. But the method appears to be a lengthy one.

In this paper we solve the general problem of the extensional vibration of a bar excited by the impact of an elastic load in a simpler way using the powerful operational method in a similar way as adopted by Ghosh (1938, 1939, 1940, 1941) in solving the general problem of pianoforte string.

In solving the problem we assume that the bar behaves like a loaded one so long as the load is in contact, and the elastic load is supposed to behave like a hard load backed by weightless spring.

We consider in section I the case when the load strikes at the free end of the bar, the other end being fixed and in section II when the load strikes as the fixed end.

Explanation of the symbols used

l = Length of the bar.

t = Variable time.

s = Variable length, measured along the length of the bar, the bar being fixed at $s=0$ and struck at $s=l$ (in sec. I). But in sec. II, the bar is fixed at $s=l$ and struck at $s=l$.

w = Displacement at any section of the bar.
w_l = Displacement at $s=1$, the struck point.
ρ = Linear density of the bar.
α = Area of the cross-section of the bar.
E_1 = Young's modulus of the material of the bar.
m = Mass of the load.
E_2 = Elastic constant of the material of the load
c = Velocity of the longitudinal wave propagation along the bar.
$\theta_1 = 2l/c$ = Period of the free vibration of the bar.
$t_n = t - n\theta_1$;
v_0 = Velocity of impact.
$J = mv_0$
u = Compression of the load.
$z = w_l + u$ = Displacement of the load.
P = Pressure exerted by the load.
D = Operator d/dt.

The differential equation of the extensional vibration is

$$\frac{d^2 w}{dt^2} = c^2 \frac{d^2 w}{ds^2} \tag{1}$$

which is equivalent to

$$\frac{d^2 w}{ds^2} = \frac{d^2}{c^2} w \tag{1.1}$$

where $c^2 = E_1\alpha/\rho$ and s is measured from the fixed end of the bar, w, the longitudinal displacement.

The elastic hammer strikes at $s=l$, and let w_l be the displacement at the struck point.

The solution of eqn. (1) is, in general, of the form,

$$w = A \sinh \frac{Ds}{c} + B \cosh \frac{Ds}{c} \tag{2}$$

The pressure P exerted by the load is given by

$$P = m\frac{d^2 Z}{dt^2} = -E_1\alpha\left(\frac{dw}{ds}\right)_{s=l} = -E_2 u \tag{3}$$

where

$$z = w_l + u \tag{4}$$

Section I

The terminal conditions are : $w=0$ at $s=0$ and $w=w_l$ at $s=l$. This reduces equation (2) to

$$w = w_l \frac{\sinh \frac{Ds}{c}}{\sinh \frac{Dl}{c}} \tag{5}$$

Now with the help of eqns. (4) and (5), eqns. (3) and (4) can be written in the form,

$$\left[mD^2 + \frac{E_1\alpha}{c} D \coth\frac{Dl}{c}\right] w_l + mD^2u = JD \tag{6}$$

and $$mD^2 w_l + (mD^2 + E_2)u = JD. \tag{7}$$

Now replacing JD by mv_0 and solving the simultaneous equations (6) and (7) for w_l and u,

we get $$w_l = \frac{1}{F(D)} v_0 \tag{8}$$

and $$u = \frac{D}{F(D)} \frac{E_1\alpha}{E_2 c} \coth\frac{Dl}{c} v_0 \tag{9}$$

$$\text{where } F(D) = D + \left(\frac{E_1\alpha}{E_2 c} D^2 + \frac{E_1\alpha}{mc}\right)\coth\frac{Dl}{c} \tag{8.1}$$

Also combining (8) and (9)

$$u = \frac{E_1\alpha}{E_2 c} \coth\frac{Dl}{c} D w_l \tag{9.1}$$

$$= \frac{E_1\alpha}{E_2 c} \coth\frac{Dl}{c} w'_l \tag{9.2}$$

In our case the load strikes at the free end at $s=1$ so that,

$$F(D) = D + \left(\frac{E_1\alpha}{E_2 c} D^2 + \frac{E_1\alpha}{mc}\right)\coth\frac{Dl}{c} = \frac{D_1D_2}{(q+p)(1-e^{-D\theta_1})}\left[1 - \left\{\frac{2(q+p)D}{D_1D_2} - 1\right\}e^{-D\theta_1}\right] \tag{10}$$

Therefore from eq. (8)

$$w_l = \frac{(q+p)(1-c^{-D\theta_1})}{D_1D_2}\left[1 - \left\{\frac{2(q+p)D}{D_1D_2} - 1\right\}e^{-D\theta_1}\right]^{-1} v_0 \tag{11}$$

where $$D_1D_2 = (D+q)(D+p) \equiv D^2 + \frac{E_2 c}{E_1\alpha} D + \frac{E_2}{m};$$

and $-q$ and $-p$ are the roots of the eq. $D_1D_2 = 0$ and are given by

$$[q, p] = \frac{E_2 c}{2E_1\alpha} \mp \tfrac{1}{2}\sqrt{\left(\frac{E_2 c}{E_1\alpha}\right)^2 - \frac{4E_2}{m}} \tag{11.1}$$

From eq. (11)

$$w_l = \frac{(q+p)}{D_1D_2}\left[1 + \left\{\frac{2(q+p)D}{D_1D_2} - 2\right\}e^{-D\theta_1}\right.$$
$$\cdot\left\{\left(\frac{2(q+p)D}{D_1D_2} - 1\right)^2 - \left(\frac{2(q+p)D}{D_1D_2} - 1\right)\right\}e^{-2D\theta_1}$$
$$\ldots + \left\{\left(\frac{2(q+p)D}{D_1D_2} - 1\right)^n - \left(\frac{2(q+p)}{D_1D_2} - 1\right)^{n-1}\right\}e^{-} \quad v_0 \tag{12.1}$$

$$=\left[\frac{(q+p)}{D_1D_2}+\left\{\frac{2(q+p)^2D}{D_1^2D_2^2}-\frac{2(q+p)}{D_1D_2}\right\}e^{-D\theta_1}\right.$$

$$+\left\{4\frac{(q+p)^3D^2}{D_1^3D_2^3}-\frac{6(q+p)^2D}{D_1^2D_2^2}+\frac{2(q+p)}{D_1D_2}\right\}e^{-D\theta_1}$$

$$+\left\{\frac{8(q+p)^4D^3}{D_1^4D_2^4}-\frac{16(q+p)^3D^2}{D_1^3D_2^3}+\frac{10(q+p)^2D}{D_1^2D_2^2}-\frac{2(q+p)}{D_1D_2}\right\}e^{-3D\theta_1}$$

$$+\ +\ +\ \dots+\left\{\frac{2^n(q+p)^{n+1}D^n}{D_1^{n+1}D_2^{n+1}}-\frac{2(n+1)(q+p)^nD^{n-1}}{D_1^nD_2^n}+-+-+\right.$$

$$+(-1)^r({}^nC_r+{}^{n-1}C_{r-1})\frac{2^{n-r}(q+p)^{n-r+1}D^{n-r}}{D_1^{n-r+1}D_2^{n-r+1}}+-+-+--$$

$$\left.\left.+(-1)^n\frac{2(q+p)}{D_1D_2}\right\}e^{-nD\theta_1}\right]v_0 \qquad (12.1)$$

$$=f_1(t)+2f_2(t_1)-2f_1(t_1)+4f_3(t_2)-6f_2(t_2)+2f_1(t_2)+8f_4(t_3)-16f_3(t_3)+10f_2(t_3)$$
$$-2f_1(t_3)+\ +\ \dots+2^nf_{n+1}(t_n)-2^{n-1}(n+1)f_n(t_n)\ \dots+(-)1^r({}^nC_r$$
$$+{}^{n-1}C_{r-1})2^{n-r}f_{n-r+1}(t_n)+\ \dots+(-1)^n2f_1(t_n) \qquad \dots (12.3)$$

The values of these functions are (Ghosh, 1938.)

Case I

When $\left(\frac{E_2c}{E_1\alpha}\right)^2>\frac{4E_2}{m}$;

$$f_1(t)=\frac{(q+p)}{D_1D_2}.v_0=v_0A\left[\frac{1}{q}(1-e^{-qt})-\frac{1}{p}(1-e^{-pt})\right] \qquad \dots (12a\ 1)$$

$$f_2(t)=\frac{(q+p)^2D}{D_1^2D_2^2}e^{-D\theta_1}.v_0=v_0A^2\left[\frac{e^{-qt_1}}{q}(1-A+qt_1)+\frac{e^{-pt_1}}{p}(1+A+pt_1)\right] \qquad (12a.2)$$

$$f_3(t_2)=\frac{(q+p)^3D^2}{D_1^3D_2^3}e^{-D\theta_1}v_0=v_0A^3\left[\frac{e^{-qt_2}}{q}\left\{\frac{3}{2}(A-A)^2+\frac{1}{2}(3A-1)qt_2-\frac{q^2t_2^2}{2!}\right\}\right.$$
$$\left.+\frac{e^{-pt_2}}{p}\left\{\frac{3}{2}(A+A)^2+\frac{1}{2}(3A+1)+\frac{p^2t_2^2}{2!}\right\}\right], \qquad \dots (12a.3)$$

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$$f_n(t)=\frac{(q+p)^nD^{n-1}}{D_1^nD_2^n}v_0$$
$$=A^nv_0\left[\sum_{r=1}^{n}(-1)^{r-1}\frac{\lfloor(n+r-1)}{\lfloor(n)\lfloor(r)}\beta^{r-1}e^{-qt}(D-q)^{n-1}\frac{t^{n-r}}{(n-r)!}\right.$$
$$\left.+(-1)^n\sum_{=1}^{n}\frac{\lfloor(n-r+1)}{\lfloor n\lfloor r}\beta^{r-1}e^{-pt}(D-p)^{n-1}\frac{t^{n-r}}{(n-r)!}1\right] \qquad (12a.n)$$

and so on :

where
$$A=\frac{q+p}{p-q}, \quad \text{and } \beta=\frac{1}{p-q}.$$

Case II

When
$$\left(\frac{E_2c}{E_1\alpha}\right)^2=\frac{4E_2}{m}\ ; \ i.e.\ \frac{4P}{m}=\frac{E_2}{E_1\alpha}$$

and
$$D_1=D_2=D+q.$$

$$f_1(t)=\frac{2q}{(D+q)^2}v_0=\frac{2v_0}{q}\left[1-e^{-qt}(1+qt)\right] \quad \dots (12b.1)$$

$$f_2(t_1)=\frac{4q^2D}{(D+q)^4}v_0e^{-D\theta_1}=4v_0\frac{e^{-qt_1}}{q}\cdot\frac{(qt_1)^3}{3!}; \quad \dots (12b.2)$$

$$f_3(t_2)=\frac{8q^3D^2}{(D+q)^6}e^{-2D\theta_1}v_0=\frac{8v_0}{q}e^{-qt_1}\left[\frac{(qt_2)^4}{4!}-\frac{(qt_2)^5}{5!}\right] \quad \dots (12b.3)$$

etc., etc. . . . . . .

$$f_n(t_{n-1})=\frac{(2q)^nD^{n-1}}{(D+q)^{2n}}e^{-(n-1)D\theta_1}v_0=\frac{2^nv_0}{q}e^{-qt_{n-1}}\left[\frac{(qt_{n-1})^{n+1}}{(n+1)!}-{}^{n-2}C_1\frac{(qt_{n-1})^{n+2}}{(n+2)!}\right.$$
$$+-+-+(-1)^{r-1}\,{}^{n-2}C_{r-1}\frac{(qt_{n-1})^{n+r}}{(n+r)!}+-+-\dots$$
$$\left.+(-1)^{n-2}\frac{(qt_{n-1})^{2n-2}}{(2n-2)!}\right] \quad \dots (12b.n)$$

Case III

When $\left(\frac{E_2c}{E_1\alpha}\right)^2<\frac{4E_2}{m}$ both p and q are complex quantities, from eqn. (11.1) $[q, p]=\mu\mp i\nu$

where,
$$\mu=\tfrac{1}{2}\frac{E_2c}{E_1\alpha} \quad \text{and } \nu=\tfrac{1}{2}\sqrt{\frac{4E_2}{m}-\left(\frac{E_2c}{E_1\alpha}\right)^2}$$

Now putting these values of q and p in eqns. 12a.1 ; 12a.2, etc. we have,

$$f_1(t)=\frac{2\mu}{\nu}v_0\left[\frac{\nu}{\mu^2+\nu^2}-\frac{1}{\sqrt{\mu^2+\nu^2}}e^{-\mu t}\sin\left(\nu t+\tan^{-1}\frac{\nu}{\mu}\right)\right] \quad (12c.1)$$

$$f_2(t_1)=\frac{2\mu^2}{\nu^3}v_0e^{-\mu t_1}\left[\sin\nu t_1-\nu t_1\cos\nu t_1\right] \quad \dots (12c.2)$$

$$f_3(t_2)=\frac{\mu^3}{\nu^5}v_0e^{-\mu t_2}\left[\sqrt{\mu^2+\nu^2}\left\{\nu^2t_2^2\sin\left(\nu t_2-\tan^{-1}\frac{\nu}{\mu}\right)+\nu t\cos\left(\nu t_2-\tan^{-1}\frac{\nu}{\mu}\right)\right\}\right.$$
$$\left.+\mu\left\{2\nu t_2\cos\nu t_2-3\sin\nu t_2\right\}\right] \quad (12c.3)$$

and so on. These are similar to those obtained by Ghosh.

Case IV

$E_2=\infty$, here $u=0$, hence, $p=\infty$, $q=+\dfrac{c}{m_1 l}$ and $A=1$, $\beta=0$.

where,
$$m_1=\frac{m}{\rho l}\,.$$

$$f_1(t)=\frac{v_0}{q}(1-e^{-qt}) \qquad \ldots \quad (12d.1)$$

$$f_2(t_1)=\frac{v_0}{q}e^{-qt_1}qt_1 \qquad \ldots \quad (12d.2)$$

$$f_3(t_2)=\frac{v_0}{q}e^{-qt_2}\left\{qt_2-\frac{q^2t_2^2}{2!}\right\} \qquad \ldots \quad (12d.3)$$

and so on :
The displacement at the struck point during contact in different intervals can be obtained easily from eqn. (12.3).

Thus, during
$$0<t<\theta_1\,;$$
$$w_l=f_1(t),$$
during
$$\theta_1<t<2\theta_1$$
$$w_l=f_1(t),\ +2f_2(t_1)-2f_1(t_1)$$
during
$$2\theta_1<t<3\theta_1$$
$$w_l=f_1(t)+2f_2(t_1)-2f_1(t_1)+4f_3(t_2)-6f_2(t_2)+2f_1(t_2)$$
and so on.

The pressure exerted by the load is numerically equal to $E_2.u$ which, by the help of eqn. (9.2), is given by

$$P=\frac{E_1\alpha}{c}\operatorname{Coth}\frac{Dl}{c}\,.\,w_l'.$$

$$=\frac{E_1\alpha}{c}\left\{1+2e^{-D\theta_1}+2e^{-2D\theta_1}+2e^{-3D\theta_1}+\ldots+\ldots\right\}w_l' \qquad \ldots \quad (13)$$

The eqn. (13) by the help of (12.3) can be written as,

$$P=\frac{E_1\alpha}{c}\Big[f_1'(t)+2f_2'(t_1)+4f_3'(t_2)-2f_2'(t_2)+8f_4'(t_3)-8f_3'(t_3)$$
$$+2f_2'(t_3)+\ldots\ldots+\Big] \qquad \ldots \quad (14)$$

So, neglecting the sign for the time being (as $P=E_2u$), we get, the expression for pressures at different intervals.

$$P_1=\frac{E_1\alpha}{c}f_1'(t) \qquad \ldots \quad (14.1)$$

$$P_2 = P_1 + \frac{2E_1\alpha}{c} f_2'(t_1) \quad \dots \quad (14.2)$$

$$P_3 = P_2 + \frac{2E_1\alpha}{c} \left[2f_3'(t_2) - f_2'(t_2) \right] \quad \dots \quad (14.3)$$

$$P_4 = P_3 + \frac{2E_1\alpha}{c} \left[4f_4'(t_3) - 4f_3'(t_3) + f_2'(t_3) \right] \quad \dots \quad (14.4)$$

and so on :

These pressures may be calculated in the different cases mentioned above.

Case I

Combining (14.1) with the first differential of (12*a*.1) we get,

$$P_1 = \rho v_0 cA(e^{-qt} - e^{-pt}) \quad \dots \quad (15.1)$$

In a like manner,

$$P_2 = P_1 + 2\rho v_0 cA^2[e^{-qt_1}(A - qt_1) - e^{-pt_1}(A + pt_1)] \quad \dots \quad (15.2)$$

$$P_3 = P_2 + 2\rho v_0 cA^2[e^{-qt_2}\{Aq^2t^2{}_2 - qt_2(3A^2 + A - 1) - 3A^3\} - e^{-pt_2}\{Ap^2t_2{}^2 - pt_2(2A + 1) + 3A^2(A + 2)\}] \quad \dots \quad (15.3)$$

and so on :

Case II

Similarly, combining (14.1) with the first differential of (12*b*.1) we get,

$$P_1 = 2\rho v_0 ce^{-qt}.qt \; ; \quad (15.4)$$

In a like manner,

$$P_2 = P_1 + 8\rho v_0 ce^{-qt_1}\left\{ \frac{(qt_1)^2}{2\,!} - \frac{(qt_1)^3}{3\,!} \right\} \quad \dots \quad (15.5)$$

$$P_3 = P_2 + 8\rho v_0 ce^{-qt_2}\left[\frac{(pt_2)^2}{2\,!} - 5.\frac{(qt_2)^3}{3\,!} + 8.\frac{(qt_2)^4}{4\,!} - 4.\frac{(qt_2)^5}{5\,!} \right] \quad \dots \quad (15.6)$$

$$P_4 = P_3 + 8\rho v_0 ce^{-pt_3}\left[\frac{(qt_3)^2}{2\,!} - 9.\frac{(qt_3)^3}{3\,!} + 32.\frac{(qt_3)^4}{4\,!} - 56.\frac{(qt_3)^5}{5\,!} + 48.\frac{(qt_3)^6}{6\,!} - 16.\frac{(qt_3)^7}{7\,!} \right] \quad \dots \quad (15.7)$$

and so on :

Case III

When $\left(\frac{E_2c}{E_1\alpha} \right)^2 < \frac{4E_2}{m}$

We get, before, combining eqn. (14.1) with the first differential of (12*c*.1)

$$P_1 = 2\rho v_0 c\mu e^{-\mu t} \sin \nu t \quad \dots \quad (15.8)$$

In a like manner,

$$P_2=P_1+4\rho v_0 c\frac{\mu^2}{\nu^3}e^{-\mu t_1}\left[\sqrt{\mu^2+\nu^2}.\nu t_1\cos\left(\nu t_1-\tan^{-1}\frac{\nu}{\mu}\right)-\mu\sin\nu t_1\right] \quad \dots \quad (15.9)$$

$$P_3=P_2+4\rho v_0 c\frac{\mu^3}{\nu^5}e^{-\mu t_2}\Bigg[(3\mu^2-\frac{\nu^4 t_2}{\mu}-2\mu\nu^2 t_2+\nu^2)\sin\nu t_2$$

$$-2(2\nu\mu^2 t_2+\mu\nu+\nu^3 t_1)\cos\nu t_2+\sqrt{\mu^2+\nu^2}\Big\{\nu^2 t_2(1-\mu t_2)\sin\left(\nu t_2-\tan^{-1}\frac{\nu}{\mu}\right)$$

$$+\nu(\nu^2 t_2^2-\mu t_2+1)\cos\left(\nu t_2-\tan^{-1}\frac{\nu}{\mu}\right)\Big\}\Bigg] \quad \dots \quad (16)$$

and so on.

Case IV

$E_2=\infty$, *i.e.*, in the case of hard load, combining (14.1) with the first differential co-efficient of (d.1),

we get,

$$P_1=\rho v_0 c e^{-qt} \quad \dots \quad (16.1)$$

In a like manner,

$$P_2=P_1+2\rho v_0 c e^{-qt_1}(1-qt_1) \quad \dots \quad (16.2)$$

$$P_3=P_2+2\rho v_0 c e^{-qt_2}(q^2t_2^2-2qt_2-3) \quad \dots \quad (16.3)$$

and so on :

Section II

Here the terminal condition at $s=0$ is $\frac{dw}{ds}=0$ for all values of t, and at $s=l$, the terminal condition is the equation of motion of the striking body.

With help of the above terminal conditions eqn. (2) becomes

$$w=w_l\frac{\cosh\frac{Ds}{c}}{\cosh\frac{Dl}{c}} \quad \dots \quad (17)$$

with the help of eqns. (4) and (17) eqns. (3) and (4) can be written in the form

$$\left[mD^2+\frac{E_1\alpha}{c}D\tanh\frac{Dl}{c}\right]w_l+mD^2u=JD \quad \dots \quad (18)$$

and

$$mD^2w_l+(mD^2+E_2)u=JD \quad \dots \quad (19)$$

Now replacing JD by mv_0 and solving the simultaneous equations (18) and (19) for w_l and u, we get,

$$w_l=\frac{1}{F(D)}v_0 \quad \dots \quad (20)$$

and

$$u=\frac{D}{F(D)}\cdot\frac{E_1\alpha}{E_2c}\tanh\frac{Dl}{c}v_0 \quad \dots \quad (21)$$

where $$F(D)=D+\left(\frac{E_1\alpha}{E_2c}D^2+\frac{E_1\alpha}{mc}\right)\tanh\frac{Dl}{c} \qquad \dots (22)$$

Also combining (20) and (21)

$$u=\frac{E_1\alpha}{E_2c}\tanh\frac{Dl}{c}Dw_l \qquad \dots (22.1)$$

$$=\frac{E_1\alpha}{E_2c}\tanh\frac{Dl}{c}w'_l \qquad \dots (22.2)$$

From eqn. (22)

$$F(D)=\frac{D_1D_2}{(q+p)(1+e^{-D\theta_1})}\left[1-\left(1-\frac{2D(q+p)}{D_1D_2}\right)e^{-D\theta_1}\right] \qquad \dots (23)$$

Therefore, from eqn. (20)

$$w_l=\frac{(q+p)(1+e^{-D\theta_1})}{D_1D_2}\left[1-\left(1-\frac{2D(q+p)}{D_1D_2}\right)e^{-D\theta_1}\right]^{-1}v_0 \qquad \dots (24)$$

where, $$D_1D_2=(D+q)(D+p)\equiv D^2+\frac{E_2c}{E_1\alpha}D+\frac{E_2}{m}$$

and $-q$, $-p$ are the roots of the eqn. $D_1D_2=0$

and are given by $$[q,\ p]=\frac{E_2c}{2E_1\alpha}\mp\frac{1}{2}\left[\left(\frac{E_2c}{E_1\alpha}\right)^2-\frac{4E_2}{m}\right]^{\frac{1}{2}}$$

From eqn. (24)

$$w_l=\frac{(q+p)}{D_1D_2}\left[1+\left\{2-\frac{2D(q+p)}{D_1D_2}\right\}e^{-D\theta_1}+\left\{\frac{4(q+p)^2D^2}{D_1^2D_2^2}-\frac{6D(q+p)}{D_1D_2}+2\right\}e^{-2D\theta_1}\right.$$
$$\left.-\left\{\frac{8(q+p)^3D^3}{D_1^3D_2^3}-\frac{16D^2(q+p)^2}{D_1^2D_2^2}+\frac{10D(q+p)}{D_1D_2}-2\right\}e^{-3D\theta_1}+\dots\right]v_0 \quad \dots (25)$$

$$=f_1(t)-2f_2(t_1)+2f_1(t_1)+4f_3(t_2)-6f_2(t_2)+2f_1(t_2)$$
$$-8f_4(t_3)+16f_3(t_3)-10f_2(t_3)+2f_1(t_3)+\dots\dots+ \qquad \dots (26)$$

The values of these functions, however, are the same as those derived in section I.

Further, the displacement equation shows, that the wave train does not return after reflection, as shown by the second term of eqn. (27) below.

The pressure exerted by the load is numerically equal to E_2u and given by eqn. (21.2) as,

$$P=\frac{E_1\alpha}{c}\tanh\frac{Dl}{c}w_l'$$

$$=\frac{E_1\alpha}{c}\left\{1-2e^{-D\theta_1}+2e^{-2D\theta_1}-2e^{-3D\theta_1}+-+-\dots\dots\right\}w_l' \qquad \dots (27)$$

Eqn. (27) with the help of eqn. (26) can be written as

$$P=\frac{E_1\alpha}{c}\left[f_1'(t_1)-2f_2'(t_1)+4f_3'(t_2)-2f_2'(t_2)+\ldots\ldots\right] \quad \ldots \quad (28)$$

The pressure equation shows that it terminates during the first interval only and can be evaluated for all the three cases with the help of the values of q and p. Thus pressure during the interval,

$$0 < t < \theta_1$$

$$P_1=\rho c v_0 A(e^{-qt}-e^{-pt}), \qquad \text{where} \qquad A=\frac{q+p}{p-q}$$

By studying the pressure functions from eqns. (14) and (15) it can be easily shown that when $E_2=\infty$, pressure increases by sudden jump in magnitude $E_1\alpha v_0/c$ at $t=0$, whence it falls down slowly to a minimum value till at $t=2l/c$ the pressure again rises suddenly in magnitude $E_1\alpha v_0/c$. This process continues with the sudden rise of pressure in magnitudes $E_1\alpha v_0/c$ at $t=0$, $2l/c$, $4l/c$, etc. till the impact terminates.

But in the case when $E_2\neq 0$, we find that this discontinuous periodic rise of pressure as obtained when $E_2=\infty$, gradually lose their sharp angularities and well-rounded humps appear instead. As the value of E_2 is diminished, the humps become less and less pronounced and we find that the duration of contact gradually increases as the hardness of the striking hammer gradually decreases.

In the second case when the load strikes at the fixed end, the pressure for $E_2=\infty$ suddenly rises by the same magnitude $E_1\alpha v_0/c$ at $t=0$ and terminates at $t=2l/c$. But when $E_2\neq 0$, pressure continuously increases, attains a maximum value and then gradually falls to zero. The duration of contact in this case is found to be greater than $2l/c$; but in case of small value of E_2 *i.e.*, for light and soft load, duration of contact is found to be less than $2l/c$. The experimental study of the problem is in progress and will be published in due course.

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